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Generalized Robertson intelligent states and squeezing for supersymmetric and shape-invariant systems: an algebraic construction

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Abstract
We obtain the Robertson–Schrödinger uncertainty relation for shape-invariant systems and construct generalized Robertson intelligent states for these systems using an algebraic approach based on the supersymmetric quantum mechanics. Using the variances of generalized quadrature operators we study the coherency and squeezing properties, evaluating their dependence on different kinds of generalizations for shape-invariant systems with potential parameters related by a translation (Pöschl–Teller potential) and by a scaling (self-similar potential).

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1. Introduction

Coherent states, known as the closest states to classical ones, were introduced by Schrödinger [1] in the early days of quantum mechanics. Based on the Heisenberg–Weyl group and applied specifically to the harmonic oscillator system, the original coherent state introduced by Schrödinger has been extended to a large number of Lie groups with square integrable representations [2, 3] and was applied to many fields of quantum theory such as quantum optics, solid state physics, astrophysics and cosmology [4–6]. The extension of coherent states for systems other than harmonic oscillator has attracted much attention for the past several years [7–13].

In quantum mechanics, coherent states can be defined in three different ways: (i) as eigenstates with complex eigenvalues of an annihilation group operator; (ii) as orbits of the ground state under the action of a unitary displacement operator and (iii) as states which minimize the Heisenberg uncertainty relation for canonical observables with equal uncertainties. These three different definitions are equivalent only in the special case of the dynamical symmetry group of the harmonic oscillator and satisfy the properties necessary for a close connection between classical and quantum formulations of a given system. Clearly, the uncertainty principle limits the precise knowledge of all physical quantities of a quantum system. In this sense, the coherent states which minimize the uncertainty product are of particular interest since they describe the quantum system as precisely as possible. In the construction of coherent states for systems other than harmonic oscillator it has been observed [14–16] that a more accurate uncertainty relation than the Heisenberg relation, known as Schrödinger–Robertson uncertainty inequality [17], may be used. When the two Hermitian operators entering in the Heisenberg uncertainty relation are noncanonical operators the result could be redundant, while the Schrödinger–Robertson uncertainty relation is not. This fact makes the Heisenberg uncertainty relation a particular case of the Schrödinger–Robertson one. The states which minimize the Schrödinger–Robertson uncertainty relation are called correlated states or Robertson intelligent states [14–16].

In supersymmetric quantum mechanics, usually studied in the context of one-dimensional systems [18], the partner Hamiltonians $\hat{H}_-$ and $\hat{H}_+$, given in the $\{|x\rangle\}$ representation by expressions

$$\hat{H}_- = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(-)}(x) = \hbar\Omega\hat{A}^\dagger\hat{A} \qquad \text{and} \qquad \hat{H}_+ = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(+)}(x) = \hbar\Omega\hat{A}\hat{A}^\dagger, \tag{1}$$

are most readily written in terms of one-dimensional operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar\Omega}}\left\{W(x) + \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}_x\right\} \qquad \text{and} \qquad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar\Omega}}\left\{W(x) - \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}_x\right\} \tag{2}$$

where $\hbar\Omega$ is a constant energy scale factor, introduced to permit working with dimensionless quantities, and the superpotential $W(x)$ is related to the partner potentials $V^{(\pm)}(x)$ via

$$V^{(\pm)}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2M}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{3}$$

In this paper, we use a *hat* to denote only the operators in the physical Hilbert space. On the other hand, note that in the $\{|x\rangle\}$ representation $f(\hat{x}) \equiv f(x)$ for any function f and $\hat{p}_x \equiv -\mathrm{i}\hbar\partial/\partial x$. A number of such pairs of Hamiltonians $\hat{H}_\pm$ share the integrability condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1), \tag{4}$$

called shape invariance [19], where the parameter a_2 of the Hamiltonian is a function of its parameter a_1 and the remainder $R(a_1)$ is independent of the dynamical variables. In the cases studied so far the parameters a_1 and a_2 are either related by a translation [3, 4] or a scaling [5, 7–10]. Although not all exactly solvable problems are shape-invariant [20], supersymmetric quantum mechanics together with the shape invariance concept, especially in its algebraic formulation [21, 22], is a powerful technique to study exactly solvable systems.

In earlier works, by using an algebraic approach, we introduced coherent states for self-similar potentials [11], a class of shape-invariant systems, and presented a possible generalization of these coherent states and their relation with the Ramanujan's integrals [12]. After that we extended this generalized formalism to all shape-invariant systems [23] and showed that the generalized coherent states then obtained satisfy the essential requirements necessary to provide the basic principles [24] embodied in Schrödinger's original idea. The purpose of this paper is to further extend the classes of Robertson intelligent states for supersymmetric and shape-invariant quantum system using an algebraic approach. The outline of the paper is the following. In section 2, we briefly recall the algebraic formulation to shape invariance and introduce the fundamental principles to construct our generalized intelligent states. In section 3, we study the coherency and squeezing properties of these states. In sections 4 and 5, we apply our general formalism for shape-invariant systems with potential parameters related for a translation and for a scaling relation, respectively. Finally, brief remarks close the paper in section 6.

2. Generalized Robertson intelligent states for shape-invariant systems

2.1. Algebraic formulation to shape-invariance

Introducing the parameter translation operator [21] T which acts in the a_n-potential parameters space $\mathcal{E}_a \equiv \{|a_n\rangle; n = 1, 2, 3, \ldots\}$ and the similarity transformation $T O(a_1) T^\dagger = O(a_2)$ that replaces a_1 with a_2 in a given operator or function O, then we can define the operators $\hat{B}_+ = \hat{A}^\dagger(a_1)T$ and $\hat{B}_- = \hat{B}_+^\dagger = T^\dagger \hat{A}(a_1)$ and we can write the Hamiltonians of equation (1) with the forms $\hat{H}_- = \hbar\Omega\hat{\mathcal{H}}_-$ and $\hat{H}_+ = \hbar\Omega T\hat{\mathcal{H}}_+ T^\dagger$, where $\hat{\mathcal{H}}_\pm = \hat{B}_\mp \hat{B}_\pm$. As a consequence of these definitions [21], the shape-invariant condition (4) can be written as the commutation relation $[\hat{B}_-, \hat{B}_+] = T^\dagger R(a_1) T \equiv R(a_0)$, where we used the identity $R(a_n) = T R(a_{n-1}) T^\dagger$, valid for any $n \in \mathbb{Z}$. This commutation relation suggests that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potential systems provided that their non-commutativity with $R(a_1)$ is taken into account. Indeed, using relations

$$R(a_n)\hat{B}_+ = \hat{B}_+ R(a_{n-1}) \qquad \text{and} \qquad R(a_n)\hat{B}_- = \hat{B}_- R(a_{n+1}) \tag{5}$$

which readily follow from the definitions of $\hat{B}_\pm$, one gets the commutation relations

$$[\hat{B}_+, R(a_0)] = \{R(a_1) - R(a_0)\}\hat{B}_+ \tag{6}$$

$$[\hat{B}_+, \{R(a_1) - R(a_0)\}\hat{B}_+] = \{R(a_2) + R(a_0)\}\hat{B}_+^2, \tag{7}$$

and so on. These infinite commutation relations and their complex conjugates together with the commutator $[\hat{B}_-, \hat{B}_+] = R(a_0)$ form an infinite-dimensional Lie algebra [21].

The ground state of the Hamiltonian $\hat{\mathcal{H}}_-$ satisfies the condition $\hat{A}|\Psi_0\rangle = 0 = \hat{B}_-|\Psi_0\rangle$. Using this fact and relations (5) it is possible to show that the nth excited eigenstate of that Hamiltonian

$$\hat{\mathcal{H}}_-|\Psi_n\rangle = e_n|\Psi_n\rangle \qquad \text{and} \qquad \hat{\mathcal{H}}_+|\Psi_n\rangle = \{e_n + R(a_0)\}|\Psi_n\rangle \tag{8}$$

has the related eigenvalues e_n given by $e_0 = 0$ and

$$e_n = \sum_{k=1}^{n} R(a_k), \qquad \text{for} \quad n \geqslant 1. \tag{9}$$

Also, it is possible to show [25] that the action of the ladder operators $\hat{B}_\pm$ on the eigenstate $|\Psi_n\rangle$ is

$$\hat{B}_+|\Psi_n\rangle = \sqrt{e_{n+1}}|\Psi_{n+1}\rangle \qquad \text{and} \qquad \hat{B}_-|\Psi_n\rangle = \sqrt{e_{n-1} + R(a_0)}|\Psi_{n-1}\rangle. \tag{10}$$

2.2. Generalized quadrature operators

From the ladder operators $\hat{B}_\pm$ we introduce two generalized quadrature operators $\hat{X}_k = \frac{1}{\sqrt{2}}(\alpha_k \hat{B}_+ + \hat{B}_- \beta_k)$ and $\hat{P}_k = \frac{\mathrm{i}}{\sqrt{2}}(\alpha_k \hat{B}_+ - \hat{B}_- \beta_k)$ which are supposed to satisfy the generalized canonical commutation relation $[\hat{X}_k, \hat{P}_k] = \mathrm{i}C_k\hat{I}$. Here, the shorthand notation $\Omega_k \equiv \Omega(a_1, a_2, a_3, \ldots)$, valid for $\Omega_k = \alpha_k, \beta_k$ or C_k, stand for arbitrary *functionals* of the potential parameters, introduced to establish a more general approach. In these conditions, if we take into account the definition of the T operator, we observe that

$$\begin{aligned} T\alpha_k T^\dagger &= T\alpha(a_1, a_2, a_3, \ldots)T^\dagger \\ &= \alpha(a_2, a_3, a_4, \ldots) \equiv \alpha_{k+1} \end{aligned} \tag{11}$$

and the same property is valid for the other functionals β_k and C_k. It is trivial to verify that the Hermitian operator properties $\hat{X}_k = \hat{X}_k^\dagger$ and $\hat{P}_k = \hat{P}_k^\dagger$ impose the conditions $\hat{B}_-\alpha_k^* = \hat{B}_-\beta_k$ and $\beta_k^*\hat{B}_+ = \alpha_k\hat{B}_+$, common solution of which requires to have $\beta_k = \alpha_k^*$. Therefore, if we take into account the properties (5) we can write

$$\begin{aligned} \hat{X}_k &= \frac{1}{\sqrt{2}}[\alpha_k\hat{B}_+ + \hat{B}_-\alpha_k^*] = \frac{1}{\sqrt{2}}[\alpha_k\hat{B}_+ + \alpha_{k-1}^*\hat{B}_-], \\ \hat{P}_k &= \frac{\mathrm{i}}{\sqrt{2}}[\alpha_k\hat{B}_+ - \hat{B}_-\alpha_k^*] = \frac{\mathrm{i}}{\sqrt{2}}[\alpha_k\hat{B}_+ - \alpha_{k-1}^*\hat{B}_-] \end{aligned} \tag{12}$$

where $\alpha_{k-1}^* = T^\dagger\alpha_k^* T$. The commutator of these operators gives $[\hat{X}_k, \hat{P}_k] = \mathrm{i}(\alpha_{k-1}^*\alpha_{k-1}\hat{B}_-\hat{B}_+ - \alpha_k\alpha_k^*\hat{B}_+\hat{B}_-)$, and to impose the canonical commutation between them we must assume that the additional condition

$$\alpha_{k-1}^*\alpha_{k-1} = \alpha_k\alpha_k^* \tag{13}$$

is satisfied. Taking into account this condition and the commutation relation between the operators $\hat{B}_\pm$ we obtain

$$[\hat{X}_k, \hat{P}_k] = \mathrm{i}\alpha_k\alpha_k^* R(a_0)\hat{I} = \mathrm{i}C_k\hat{I}. \tag{14}$$

Consequently, the generalized Hamiltonian $\hat{H}_k = \frac{1}{2}\left(\hat{X}_k^2 + \hat{P}_k^2\right)$ associated with the quadrature operators has the form

$$\hat{H}_k = \alpha_k\alpha_k^*\left\{\hat{\mathcal{H}}_- + \tfrac{1}{2}R(a_0)\hat{I}\right\}, \tag{15}$$

which differs from the Hamiltonian $\hat{\mathcal{H}}_-$ only in the shift term $\frac{1}{2}R(a_0)$ and in a constant scale factor, since the canonical relation condition (13) implies that the term $\alpha_k\alpha_k^*$ must be a constant, independent of the potential parameters a_n.

2.3. Robertson–Schrödinger uncertainty relation

Consider the state

$$|\Phi\rangle = (\hat{X}_k - \langle\hat{X}_k\rangle)|\Psi\rangle + \mathrm{i}(\hat{P}_k - \langle\hat{P}_k\rangle)|\Psi\rangle\lambda \tag{16}$$

where $\lambda \equiv \lambda(a_1, a_2, a_3, \ldots)$ is a *complex function* of the potential parameters and the notation $\langle\hat{O}\rangle = \langle\Psi|\hat{O}|\Psi\rangle$ stands for the expectation value of a given observable $\hat{O}$ in the state $|\Psi\rangle$ of the quantum system. Since for all λ we must have $\langle\Phi|\Phi\rangle \geqslant 0$, then with the definition (16) we obtain the relation

$$\begin{aligned} &\langle\Psi|(\hat{X}_k - \langle\hat{X}_k\rangle)^2|\Psi\rangle + \lambda^*\langle\Psi|(\hat{P}_k - \langle\hat{P}_k\rangle)^2|\Psi\rangle\lambda + \mathrm{i}\{\langle\Psi|(\hat{X}_k - \langle\hat{X}_k\rangle)(\hat{P}_k - \langle\hat{P}_k\rangle)|\Psi\rangle\lambda \\ &\quad - \lambda^*\langle\Psi|(\hat{P}_k - \langle\hat{P}_k\rangle)(\hat{X}_k - \langle\hat{X}_k\rangle)|\Psi\rangle\} \geqslant 0. \end{aligned} \tag{17}$$

By using the complex form $\lambda = \lambda^{(R)} + \mathrm{i}\lambda^{(I)}$ and observing that λ commutes with the scalar terms $\langle\hat{P}_k\rangle$ and $\langle\hat{X}_k\rangle$, we can write relation (17) as

$$\mathcal{I}_k(\lambda^{(R)}, \lambda^{(I)}) \equiv (\Delta\hat{X}_k)^2 + (\Delta\hat{P}_k)^2\{[\lambda^{(R)}]^2 + [\lambda^{(I)}]^2\} - \langle\hat{F}_k\rangle\lambda^{(I)} + \mathrm{i}\langle\hat{G}_k\rangle\lambda^{(R)} \geqslant 0 \tag{18}$$

where we used the variance definition of a given observable $\hat{O}$ in the state $|\Psi\rangle$

$$\sigma_O \equiv (\Delta\hat{O})^2 \equiv \langle\Psi|(\hat{O} - \langle\hat{O}\rangle)^2|\Psi\rangle = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2, \tag{19}$$

the expectation value of the anti-Hermitian canonical commutator

$$\hat{G}_k \equiv [\hat{X}_k - \langle\hat{X}_k\rangle\hat{I}, \hat{P}_k - \langle\hat{P}_k\rangle\hat{I}] = [\hat{X}_k, \hat{P}_k] = \mathrm{i}\alpha_k\alpha_k^* R(a_0)\hat{I} = \mathrm{i}C_k\hat{I} \tag{20}$$

and the expectation value of the Hermitian anticommutator operator of the generalized quadrature operators

$$\hat{F}_k \equiv \{\hat{X}_k - \langle\hat{X}_k\rangle\hat{I}, \hat{P}_k - \langle\hat{P}_k\rangle\hat{I}\}. \tag{21}$$

By virtue of the positive value of the variances of $\hat{X}_k$ and $\hat{P}_k$, the function $\mathcal{I}_k(x_j, y_j)$ represents the upper part of the hyperboloid of two sheets function, minimum value of which can be obtained applying directly the minimization conditions

$$\frac{\partial \mathcal{I}_k(\lambda^{(R)}, \lambda^{(I)})}{\partial \lambda^{(R)}} = 2(\Delta \hat{P}_k)^2 \lambda^{(R)} + \mathrm{i}\langle \hat{G}_k \rangle = 0 \qquad \text{and}$$
$$\frac{\partial \mathcal{I}_k(\lambda^{(R)}, \lambda^{(I)})}{\partial \lambda^{(I)}} = 2(\Delta \hat{P}_k)^2 \lambda^{(I)} - \langle \hat{F}_k \rangle = 0. \tag{22}$$

Solving this system of equations we get the minimum point coordinates

$$\lambda^{(R)} \to \lambda_k^{(R)} = -\frac{\mathrm{i}\langle \hat{G}_k \rangle}{2(\Delta \hat{P}_k)^2} \qquad \text{and} \qquad \lambda^{(I)} \to \lambda_k^{(I)} = \frac{\langle \hat{F}_k \rangle}{2(\Delta \hat{P}_k)^2}, \tag{23}$$

and using them in (18) we obtain the Robertson–Schrödinger uncertainty relation [17]

$$(\Delta \hat{X}_k)^2 (\Delta \hat{P}_k)^2 \geqslant \tfrac{1}{4}\{\langle \hat{F}_k \rangle^2 - \langle \hat{G}_k \rangle^2\}. \tag{24}$$

The expectation value of the operator $\hat{F}_k$ can be explicitly written as

$$\langle \hat{F}_k \rangle = \langle \hat{X}_k \hat{P}_k \rangle + \langle \hat{P}_k \hat{X}_k \rangle - 2\langle \hat{X}_k \rangle \langle \hat{P}_k \rangle = 2\sigma_{XP}^{(k)} \tag{25}$$

where $\sigma_{XP}^{(k)}$ is the covariance of the generalized quadrature operators $\hat{X}_k$ and $\hat{P}_k$ in the state $|\Psi\rangle$. The covariance gives the measure of the correlation between two observables. Using (20) and (25) in (24), we obtain the Robertson–Schrödinger uncertainty relation applied for shape-invariant systems in terms of the variance and covariance of the quadrature operators

$$\sigma_X^{(k)} \sigma_P^{(k)} - \sigma_{XP}^{(k)} \geqslant \tfrac{1}{4}\{\alpha_k \alpha_k^* R(a_0)\}^2, \tag{26}$$

which can also be written in the form

$$\sigma_X^{(k)} \sigma_P^{(k)} \geqslant \left\{ \frac{\alpha_k \alpha_k^* R(a_0)}{4\left(1 - r_{XP}^{(k)}\right)} \right\}^2 \qquad \text{where} \quad r_{XP}^{(k)} = \frac{\sigma_{XP}^{(k)}}{\sqrt{\sigma_X^{(k)} \sigma_P^{(k)}}} \tag{27}$$

gives the *correlation coefficient* between the generalized quadrature operators $\hat{X}_k$ and $\hat{P}_k$.

When the operators $\hat{X}_k$ and $\hat{P}_k$ are uncorrelated we have $\langle \hat{F}_k \rangle = \sigma_{XP}^{(k)} = 0$ and so we regain the standard Heisenberg uncertainty principle applied for shape-invariant systems. On the other hand, for two noncommuting Hermitian operators $\hat{Y}$ and $\hat{O}$ such as $\langle \hat{D} \rangle \equiv \langle [\hat{Y}, \hat{O}] \rangle = 0$, unlike those of the Robertson–Schrödinger uncertainty relation, the Heisenberg uncertainty relation applied to these operators is redundant since it does not determine a nontrivial lower bound on the product of the operators uncertainties. In this sense, we can say that the Heisenberg uncertainty relation is a particular case of the general Robertson–Schrödinger uncertainty relation (24).

2.4. Generalized Robertson intelligent states

2.4.1. Definition. When the equality in (18) is realized and the Robertson–Schrödinger uncertainty relation (24) assumes its minimum value, the state $|\Psi\rangle \to |\lambda_k, w_k\rangle_{\mathrm{i}}$ introduced in (16) that minimizes the relation is called Robertson intelligent state. For shape-invariant systems, with the minimum point coordinates (23) and the definition (16) we can show that these states must be solution of the eigenvalue equation $|\Phi\rangle_{\min} = 0$ which reads

$$\hat{X}_k |\lambda_k, w_k\rangle_{\mathrm{i}} + \mathrm{i}\hat{P}_k |\lambda_k, w_k\rangle_{\mathrm{i}} \lambda_k = w_k |\lambda_k, w_k\rangle_{\mathrm{i}} \qquad \text{with} \quad \lambda_k = \frac{1}{2\sigma_P^{(k)}}\left\{\alpha_k \alpha_k^* R(a_0) + 2\mathrm{i}\sigma_{XP}^{(k)}\right\},$$
$$w_k = \langle \hat{X}_k \rangle + \mathrm{i}\lambda_k \langle \hat{P}_k \rangle \tag{28}$$

and the expectation values are now obtained with $\langle \hat{O} \rangle = {}_{\rm i}\langle \lambda_k, w_k | \hat{O} | \lambda_k, w_k \rangle_{\rm i}$. For quadrature operators $\langle \hat{G}_k \rangle \neq 0$, and thus $\lambda_k^{(R)} \neq 0$ also. Therefore, once we know $w_k = w_k^{(R)} + {\rm i} w_k^{(I)}$ and λ_k, the relations in (28) may be inverted to give

$$\langle \hat{X}_k \rangle = w_k^{(R)} + \left(\frac{\lambda_k^{(I)}}{\lambda_k^{(R)}} \right) w_k^{(I)} \qquad \text{and} \qquad \langle \hat{P}_k \rangle = \frac{w_k^{(I)}}{\lambda_k^{(R)}}. \tag{29}$$

Note that with the definitions (28), the properties (5) and assuming that $\lambda_{k+1} = T \lambda_k T^\dagger \neq -1$, the eigenvalue equation for the intelligent states can also be rewritten in terms of the ladder operators $\hat{B}_\pm$ as

$$\hat{B}_- |\lambda_k, w_k\rangle_{\rm i} = \left\{ \left[\frac{\alpha_k (\lambda_{k+1} - 1)}{\alpha_{k-1}^* (\lambda_{k+1} + 1)} \right] \hat{B}_+ + \frac{\sqrt{2} w_k}{\alpha_{k-1}^* (\lambda_{k+1} + 1)} \right\} |\lambda_k, w_k\rangle_{\rm i}. \tag{30}$$

2.4.2. Construction. The operator $\hat{B}_-$ does not have a left inverse in the Hilbert space $\mathcal{E}$ of the eigenstates $\{|\Psi_n\rangle, n = 0, 1, 2, \ldots\}$ of the Hamiltonian $\hat{\mathcal{H}}_-$. However, a right inverse for $\hat{B}_-$ $\{\hat{B}_- \hat{B}_-^{-1} = \hat{I}\}$ can be defined. This fact makes possible to introduce the state

$$|b, c\rangle = \sum_{n=0}^{\infty} \hat{K}^n |\Psi_0\rangle \qquad \text{with} \quad \hat{K} = \hat{B}_-^{-1} (b \hat{B}_+ + c) \tag{31}$$

where the complex factors $b \equiv b(a_1, a_2, a_3, \ldots)$ and $c \equiv c(a_1, a_2, a_3, \ldots)$ can depend on the potential parameters. From the definition (31) it follows that

$$\begin{aligned}
\hat{B}_- |b, c\rangle &= \hat{B}_- \{ |\Psi_0\rangle + \hat{B}_-^{-1} (b \hat{B}_+ + c) |\Psi_0\rangle + [\hat{B}_-^{-1} (b \hat{B}_+ + c)]^2 |\Psi_0\rangle + \cdots \} \\
&= \hat{B}_- |\Psi_0\rangle + \hat{B}_- \hat{B}_-^{-1} (b \hat{B}_+ + c) |\Psi_0\rangle + \hat{B}_- [\hat{B}_-^{-1} (b \hat{B}_+ + c)]^2 |\Psi_0\rangle + \cdots \\
&= 0 + (b \hat{B}_+ + c) |\Psi_0\rangle + (b \hat{B}_+ + c) [\hat{B}_-^{-1} (b \hat{B}_+ + c)] |\Psi_0\rangle \\
&\quad + (b \hat{B}_+ + c) [\hat{B}_-^{-1} (b \hat{B}_+ + c)]^2 |\Psi_0\rangle \cdots \\
&= (b \hat{B}_+ + c) \{ |\Psi_0\rangle + \hat{B}_-^{-1} (b \hat{B}_+ + c) |\Psi_0\rangle + [\hat{B}_-^{-1} (b \hat{B}_+ + c)]^2 |\Psi_0\rangle + \cdots \} \\
&= (b \hat{B}_+ + c) \sum_{n=0}^{\infty} \{ \hat{B}_-^{-1} (b \hat{B}_+ + c) \}^n |\Psi_0\rangle
\end{aligned}$$

that is

$$\hat{B}_- |b, c\rangle = (b \hat{B}_+ + c) |b, c\rangle. \tag{32}$$

Comparing the eigenvalues equation (30) which defines the Robertson intelligent states with equation (32) above we conclude that if we identify the arbitrary complex factors as

$$b \to b_k \equiv \frac{\alpha_k (\lambda_{k+1} - 1)}{\alpha_{k-1}^* (\lambda_{k+1} + 1)} \qquad \text{and} \qquad c \to c_k \equiv \frac{\sqrt{2} w_k}{\alpha_{k-1}^* (\lambda_{k+1} + 1)} \tag{33}$$

then the state

$$|\alpha_k, \lambda_k, w_k\rangle_{\rm i} = \sum_{n=0}^{\infty} \hat{K}_k^n |\Psi_0\rangle \qquad \text{with} \quad \hat{K}_k = \hat{B}_-^{-1} (b_k \hat{B}_+ + c_k) \tag{34}$$

is a generalized Robertson intelligent state for shape-invariant potential systems.

2.4.3. Generalized Glauber form. In the same way, the operator $\hat{\mathcal{H}}_-$ does not have an inverse in the Hilbert space $\mathcal{E}$, but the operator $\hat{\mathcal{H}}_-^{-1}\hat{B}_+ = \hat{B}_-^{-1}$ does. Therefore, if we define the Hermitian conjugate operators $\hat{Q} = \hat{B}_-\hat{\mathcal{H}}_-^{-1/2}$ and $\hat{Q}^\dagger = \hat{\mathcal{H}}_-^{-1/2}\hat{B}_+$ [11] we can easily show that $\hat{B}_-^{-1} = \hat{\mathcal{H}}_-^{-1/2}\hat{Q}^\dagger$ and the normalized form of the nth excited eigenstate of $\hat{\mathcal{H}}_-$ can be obtained by $|\Psi_n\rangle = (\hat{Q}^\dagger)^n|\Psi_0\rangle$. With this relation and the eigenvalue equations (8) we can show that the action of the operator $\hat{B}_-^{-1}$ on the state $|\Psi_n\rangle$ of the Hilbert space $\mathcal{E}$ is given by

$$\hat{B}_-^{-1}|\Psi_n\rangle = \hat{\mathcal{H}}_-^{-1/2}\hat{Q}^\dagger(\hat{Q}^\dagger)^n|\Psi_0\rangle = \hat{\mathcal{H}}_-^{-1/2}|\Psi_{n+1}\rangle = \frac{1}{\sqrt{e_{n+1}}}|\Psi_{n+1}\rangle. \tag{35}$$

Therefore using (35), the action of the $\hat{B}_+$ operator on these states (10), the translation properties (5) and the definition (34) we find that

$$\hat{K}_k|\Psi_n\rangle = b_{k+1}\sqrt{\frac{e_{n+2}-e_1}{e_{n+2}-e_0}}|\Psi_{n+2}\rangle + \frac{c_{k+1}}{\sqrt{e_{n+1}}}|\Psi_{n+1}\rangle. \tag{36}$$

Using this relation and the translation properties (5) in the expansion (34), after a long calculation, it is possible to show that we can write

$$|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}} = \sum_{n=0}^{\infty} C_{kn}|\Psi_n\rangle \tag{37}$$

where $C_{k0} = 1$ and the expansion coefficients for $n > 0$ can be obtained recursively by

$$C_{kn} = C_{kn}^{(0)}\left\{1 + \sum_{j=1}^{[n/2]} C_{kn}^{(j)}\right\}. \tag{38}$$

In this expression the symbol $[n/2]$ stands for the integer part of $n/2$ and the auxiliary coefficients are given by

$$C_{kn}^{(0)} = \prod_{s=0}^{n-1}\left(\frac{c_{k+s+1}}{\sqrt{e_n - e_s}}\right) \tag{39}$$

$$C_{kn}^{(1)} = \sum_{s=1}^{n-1} d_{ns}^{(k)}, \qquad \text{with} \quad d_{ns}^{(k)} = \frac{b_{k+s+1}(e_n - e_s)}{c_{k+s+1}c_{k+s+2}} \tag{40}$$

$$C_{kn}^{(2)} = \sum_{r=1}^{n-3}\sum_{s=1}^{r} d_{ns}^{(k)} d_{n,n+s-r-1}^{(k)} \tag{41}$$

$$C_{kn}^{(3)} = \sum_{r=1}^{n-5}\sum_{s=1}^{r} d_{ns}^{(k)} d_{n,n+s-r-1}^{(k)} d_{n,n+s-r-3}^{(k)} \tag{42}$$

$$C_{kn}^{(4)} = \sum_{r=1}^{n-7}\sum_{s=1}^{r} d_{ns}^{(k)} d_{n,n+s-r-1}^{(k)} d_{n,n+s-r-3}^{(k)} d_{n,n+s-r-5}^{(k)} \tag{43}$$

$$\vdots \quad \vdots \qquad\qquad \vdots$$

$$C_{kn}^{(j)} = \sum_{r=1}^{n-2j+1}\sum_{s=1}^{r} \underbrace{d_{ns}^{(k)} d_{n,n+s-r-1}^{(k)} d_{n,n+s-r-3}^{(k)} \cdots d_{n,n+s-r-2j+3}^{(k)}}_{\text{product of } j \text{ coefficients } d_{nm}^{(k)}} \tag{44}$$

where $b_{k+j} = \{T\}^j b_k \{T^\dagger\}^j$ and $c_{k+j} = \{T\}^j c_k \{T^\dagger\}^j$. The expansion (37) of the intelligent state $|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}}$ in the basis of the eigenstates $\{|\Psi_n\rangle, n = 0, 1, 2, \ldots\}$ of the Hamiltonian $\hat{\mathcal{H}}_-$ gives its generalized Glauber's form [4].

3. Generalized coherent and squeezed states for shape-invariant systems

3.1. Squeezing effects of the generalized Robertson intelligent states

From the minimum point coordinates (23) and the Robertson–Schrödinger uncertainty relation (26) we can show that the quadrature operators' variances and covariance factors in the intelligent state (34) are related by

$$\sigma_P^{(k)} = \frac{1}{2\lambda_k^{(R)}}\{\alpha_k\alpha_k^* R(a_0)\} = \frac{\Delta_k^{(\mathrm{R})}}{|\lambda_k|}, \qquad \sigma_X^{(k)} = |\lambda_k|^2\,\sigma_P^{(k)} = |\lambda_k|\,\Delta_k^{(\mathrm{R})} \qquad \text{and}$$
$$\sigma_{XP}^{(k)} = \lambda_k^{(I)}\sigma_P^{(k)} \tag{45}$$

where $\Delta_k^{(\mathrm{R})} = \frac{1}{2}\sqrt{\{\alpha_k\alpha_k^* R(a_0)\}^2 + 4\left\{\sigma_{XP}^{(k)}\right\}^2}$. As we have already mentioned, for quadrature operators $\lambda_k^{(R)} \neq 0$. Note that the relations in (45) make clear that the intelligent state which satisfies the eigenvalue equation (30) presents squeezed effects when $|\lambda_k| \neq 1$. Squeezed states, characterized when the variance in one of the quadrature operators is amplified while the variance in the other quadrature operator is deamplified, have attracted due attention in last decades. Taking into account that the uncertainty principle limits the precise knowledge of all physical quantities in a quantum system, we note that squeezed states are the tools used in a quantum engineering approach to *beat* the uncertainty principle in the problems of coding and transmitting informations by optical means. Because of this property the squeezed states have found potential applications in optics communication, detection of weak signals, atomic and molecular physics and quantum physics in general [26–30].

In our case, as a consequence of relations (45), we conclude that

- if $|\lambda_k| < 1$ then $\sigma_X^{(k)} < \Delta_k^{(\mathrm{R})} < \sigma_P^{(k)}$ and thus $|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}}$ will be a *X-squeezed state*;
- if $|\lambda_k| > 1$ then $\sigma_X^{(k)} > \Delta_k^{(\mathrm{R})} > \sigma_P^{(k)}$ and thus $|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}}$ will be a *P-squeezed state*.

On the other hand, after a little moment's thought one comes to the conclusion that writing the expression for the generalized intelligent state (34) as

$$|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}} = \hat{G}_k|\Psi_0\rangle, \qquad \text{where} \quad \hat{G}_k \equiv \sum_{n=0}^{\infty}\hat{K}_k^n = \sum_{n=0}^{\infty}\{\hat{B}_-^{-1}(b_k\hat{B}_+ + c_k)\}^n, \tag{46}$$

it is possible to identify the operator $\hat{G}_k$ as the shape-invariant generalization of the product of the displacement $\hat{D}$ with the squeezing $\hat{S}$ unitary operators, defined in optics for the harmonic oscillator case. We shall make it clear in the next sections that the expansion terms involving the $(\hat{B}_-^{-1})^n$ operator are related with the generalization of the unitary displacement operator $\hat{D}$ while those depending on the operators product $(\hat{B}_-^{-1}\hat{B}_+)^n$ are basically related with the generalization of the unitary squeezing operator $\hat{S}$. In this sense, when we make the factors $b_k = 0$ and $c_k \neq 0$ we have *generalized purely coherent states* while for $b_k \neq 0$ and $c_k = 0$ we construct *generalized purely squeezed states* for shape-invariant systems.

3.2. Generalized purely coherent states

Rewriting the eigenvalue equation (30) which defines the Robertson intelligent states as

$$\left\{\hat{B}_- - \left[\frac{\alpha_k(\lambda_{k+1}-1)}{\alpha_{k-1}^*(\lambda_{k+1}+1)}\right]\hat{B}_+\right\}|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}} = \left\{\frac{\sqrt{2}w_k}{\alpha_{k-1}^*(\lambda_{k+1}+1)}\right\}|\alpha_k, \lambda_k, w_k\rangle_{\mathrm{i}} \tag{47}$$

we verify that when $\lambda_{k+1} = \lambda_{k+1}^{(R)} = 1 = \lambda_k^{(R)} = \lambda_k$ this equation reduces to the form

$$\hat{B}_-|z; a_k\rangle_{\mathrm{c}} = z\mathcal{Z}_{k-1}|z; a_k\rangle_{\mathrm{c}}, \qquad z, \mathcal{Z}_{k-1} \in \mathbb{C} \tag{48}$$

since we identify $|\alpha_k, 1, w_k\rangle_{\rm i} \rightarrow |z; a_k\rangle_{\rm c}$ and $c_k = w_k/(\sqrt{2}\alpha^*_{k-1}) \rightarrow z\mathcal{Z}_{k-1}$ where $\mathcal{Z}_{k-1} = T^\dagger \mathcal{Z}_k T$ with $\mathcal{Z}_k \equiv \mathcal{Z}(a_1, a_2, a_3, \ldots)$. If we take the value of λ_k to be unity in the expression (33), together with the property (5), in the expansion (34) we obtain

$$|z; a_k\rangle_{\rm c} = \sum_{n=0}^{\infty} \{c_{k+1}\hat{B}_-^{-1}\}^n |\Psi_0\rangle = \sum_{n=0}^{\infty} \{z\mathcal{Z}_k\hat{B}_-^{-1}\}^n |\Psi_0\rangle \tag{49}$$

with

$$c_{k+1} = T\left\{\frac{w_k}{\sqrt{2}\alpha^*_{k-1}}\right\}T^\dagger = \frac{w_{k+1}}{\sqrt{2}\alpha^*_k} \equiv z\mathcal{Z}_k. \tag{50}$$

Expression (49) gives the generalized coherent state introduced in [23]. As shown in this reference, the generalized coherent state $|z; a_k\rangle_{\rm c}$ satisfies the essential requirements necessary to provide the basic principles embodied in Schrödinger's original idea. Note that with $\lambda_k = 1$ the b_k-factors introduced in (33) are null which makes null also the expansion coefficient factors $d^{(k)}_{ns}$ defined in (40). Because of this fact all the $C^{(j)}_{kn}$-Glauber expansion coefficients components in (38) with $j > 0$ disappear which makes $C_{kn} = C^{(0)}_{kn}$. In these conditions, with the use of relations (9), (50) and (39), it is easy to show that (37) becomes

$$|z; a_k\rangle_{\rm c} = \sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n(a_k)}\right\}|\Psi_n\rangle, \tag{51}$$

with $h_0(a_k) = 1$ and

$$h_n(a_k) = \prod_{s=0}^{n-1}\frac{\sqrt{e_n - e_s}}{\mathcal{Z}_{k+s}} = \prod_{s=1}^{n}\left\{\sqrt{\sum_{j=s}^{n} R(a_j)}\Big/\mathcal{Z}_{k+s-1}\right\}, \qquad \text{for} \quad n \geqslant 1 \tag{52}$$

where $\mathcal{Z}_{k+s} = \{T\}^s\mathcal{Z}_k\{T^\dagger\}^s$. As shown in [23], equations (51) and (52) give the Glauber form of the generalized coherent states for shape-invariant systems. Applications for some shape-invariant systems with the potential parameters a_n related for a translation and a scaling are presented in that reference.

In the special case of the harmonic oscillator, the simplest among the shape-invariant potential systems, the parameters are related by $a_1 = a_2 = \cdots = a_n$ which imply that we have $\mathcal{Z}_k = \text{cte.} = 1$. On the other hand, in such a case $|\Psi_n\rangle \rightarrow |n\rangle$ is an element of the Fock space $\mathcal{F} \equiv \{|n\rangle, n = 0, 1, 2, \ldots\}$ and, by definition presented, $\hat{B}_-^{-1} \rightarrow \hat{a}^{-1} \equiv (\hat{a}^\dagger\hat{a})^{-1}\hat{a}^\dagger$. Therefore, using these results in (35) we find that $\hat{a}^{-1}|n\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle$ and thus (49) assumes the form

$$|z\rangle_{\rm c} = \sum_{n=0}^{\infty}\{z\hat{a}^{-1}\}^n|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = \exp\{z\hat{a}^\dagger\}|0\rangle. \tag{53}$$

Looking at the middle terms in equation (53) we recognize the usual expression of the nonnormalized coherent state for harmonic oscillator potential systems [31]. On the other hand, comparing the result (53) with the displacement operator definition $\hat{D}(z) \equiv \exp\{z\hat{a}^\dagger - z^*\hat{a}\}$ and remembering that the term $z^*\hat{a}$ in the exponential argument of this expression, which assures the unitary character of the operator, is related with the normalization condition of the state $|z\rangle_{\rm c}$, then we conclude that the expansion terms in (34) involving the operator $\hat{B}_-^{-1}$ alone are related with the generalization of the displacement operator $\hat{D}(z)$ for shape-invariant systems.

We are now in a position to successively examine the behaviour of the variances $\sigma^{(k)}_X$ and $\sigma^{(k)}_P$ of the quadrature operators computed on the generalized purely coherent state $|z; a_k\rangle_{\rm c}$.

Using the coherent state eigenvalue equation (48) and the quadrature operators definition we can show, after some calculation, that

$$\begin{aligned}\sigma_X^{(k)} &= \left\{{}_{\rm c}\langle z; a_k|\hat{X}_k^2|z; a_k\rangle_{\rm c} - {}_{\rm c}\langle z; a_k|\hat{X}_k|z; a_k\rangle_{\rm c}^2\right\}\Big/{}_{\rm c}\langle z; a_k|z; a_k\rangle_{\rm c} \\ &= \left\{1 - 2\delta_k^{(c)}(z)\right\}\Delta_k^{(\rm H)}\end{aligned} \tag{54}$$

and

$$\begin{aligned}\sigma_P^{(k)} &= \left\{{}_{\rm c}\langle z; a_k|\hat{P}_k^2|z; a_k\rangle_{\rm c} - {}_{\rm c}\langle z; a_k|\hat{P}_k|z; a_k\rangle_{\rm c}^2\right\}\Big/{}_{\rm c}\langle z; a_k|z; a_k\rangle_{\rm c} \\ &= \left\{1 + 2\delta_k^{(c)}(z)\right\}\Delta_k^{(\rm H)},\end{aligned} \tag{55}$$

where

$$\delta_k^{(c)}(z) = \mathrm{Re}\left\{\frac{z^2\mathcal{Z}_{k-1}}{\alpha_k R(a_0)}[\alpha_k^*\mathcal{Z}_{k-1} - \alpha_{k+1}^*\mathcal{Z}_{k-2}]\right\} \tag{56}$$

and $\Delta_k^{(\rm H)} = \frac{1}{2}\alpha_k\alpha_k^* R(a_0)$ is the uncorrelated reference value related with the Heisenberg uncertainty relation. Since in general the factor $\delta_k^{(c)}(z) \neq 0$, the immediate conclusion of these results is that our generalized approach introduces, at first, squeezing effects on the purely coherent states when compared with the uncorrelated Heisenberg result $\Delta_k^{(\rm H)}$.

3.3. Generalized purely squeezed states

Another particular and interesting case of the Robertson intelligent states happens when we assume $w_k = 0$ in (47) which thus reduces to the form

$$\{\hat{B}_- - z\mathcal{Z}_{k-1}\hat{B}_+\}|z, a_k\rangle_{\rm s} = 0, \qquad z, \mathcal{Z}_{k-1} \in \mathbb{C} \tag{57}$$

since we identify $|\alpha_k, \lambda_k, 0\rangle_{\rm i} \to |z; a_k\rangle_{\rm s}$ and $b_k = \alpha_k(\lambda_{k+1} - 1)/[\alpha_{k-1}^*(\lambda_{k+1} + 1)] \to z\mathcal{Z}_{k-1}$ where $\mathcal{Z}_{k-1} = T^\dagger\mathcal{Z}_k T$. Keeping in mind the ladder character of the operators $\hat{B}_\pm$ we can recognize (57) as the shape-invariant generalization of the eigenvalue equation which defines purely squeezed states for harmonic oscillator systems [32, 33]. Taking the null value of w_k in expressions (33) and using it, together with property (5), in expansion (34), we obtain

$$|z; a_k\rangle_{\rm s} = \sum_{n=0}^{\infty}\{b_{k+1}\hat{B}_-^{-1}\hat{B}_+\}^n|\Psi_0\rangle = \sum_{n=0}^{\infty}\{z\mathcal{Z}_k\hat{B}_-^{-1}\hat{B}_+\}^n|\Psi_0\rangle \tag{58}$$

with

$$b_{k+1} = T\left\{\frac{\alpha_k(\lambda_{k+1} - 1)}{\alpha_{k-1}^*(\lambda_{k+1} + 1)}\right\}T^\dagger \equiv z\mathcal{Z}_k. \tag{59}$$

Note that with $w_k = 0$ the c_k factors introduced in (33) are null also. Using this fact in (36) and taking into account relation (59) and the translation properties (5) in expansion (34) it is possible to get, after some calculations, the Glauber form of the purely squeezed state

$$|z; a_k\rangle_{\rm s} = \sum_{n=0}^{\infty}\left\{\frac{z^n}{h_{2n}(a_k)}\right\}|\Psi_{2n}\rangle, \tag{60}$$

where $h_0(a_k) = 1$ and

$$\begin{aligned}h_{2n}(a_k) &= \prod_{s=0}^{n-1}\left\{\sqrt{\left[\frac{e_{2n} - e_{2s}}{e_{2n} - e_{2s+1}}\right]}\Big/\mathcal{Z}_{k+2s}\right\} \\ &= \prod_{s=1}^{n}\left\{\sqrt{\left[\sum_{j=2s+1}^{2n}R(a_j)\Big/\sum_{j=2s}^{2n}R(a_j)\right]}\Big/\mathcal{Z}_{k+2s-2}\right\}, \qquad n \geqslant 1\end{aligned} \tag{61}$$

with $\mathcal{Z}_{k+2s} = \{T\}^{2s}\mathcal{Z}_k\{T^\dagger\}^{2s}$.

In the special case of the harmonic oscillator $a_1 = a_2 = \cdots = a_n$ which implies to have $\mathcal{Z}_k = \text{cte.} = 1$. By definition presented, we have $\hat{B}_-^{-1}\hat{B}_+ \to \hat{a}^{-1}\hat{a}^\dagger \equiv (\hat{a}^\dagger\hat{a})^{-1}\hat{a}^{\dagger 2}$ and, with relations (10) and (35), we find that its action on the Fock states reads $\hat{a}^{-1}\hat{a}^\dagger|n\rangle = \sqrt{\frac{n+1}{n+2}}|n+2\rangle$. Therefore using this result in (58) one has

$$|z\rangle_{\text{s}} = \sum_{n=0}^{\infty}\{z\hat{a}^{-1}\hat{a}^\dagger\}^n|0\rangle = \sum_{n=0}^{\infty}\sqrt{\frac{(2n-1)!!}{(2n)!!}}z^n|2n\rangle = \exp\left\{\frac{1}{2}z\hat{a}^{\dagger 2}\right\}|0\rangle. \tag{62}$$

We recognize in the middle terms of (62) the usual expression of the nonnormalized squeezed state for harmonic oscillator potential systems [32]. Comparing (62) with the squeezing operator definition $\hat{S}(z) \equiv \exp\left\{\frac{1}{2}(z\hat{a}^{\dagger 2} - z^*\hat{a}^2)\right\}$ and taking into account that the term $\frac{1}{2}z^*\hat{a}^2$ in the exponential argument of this expression, which assures the unitary character of $\hat{S}(z)$, is related with the normalization condition of the state $|z\rangle_{\text{s}}$, then we conclude that the expansion terms involving the operators product $\hat{B}_-^{-1}\hat{B}_+$ alone in (34) are related with the generalization of the squeezing operator $\hat{S}(z)$ for shape-invariant systems.

The squeezing property of the state $|z; a_k\rangle_{\text{s}}$ can be evaluated calculating the variances of the quadrature operators $\sigma_X^{(k)}$ and $\sigma_P^{(k)}$ in this state. To proceed first, we note that by relations (10) and definitions (12) we find that $\langle\hat{X}_k\rangle = {}_{\text{s}}\langle z; a_k|\hat{X}_k|z; a_k\rangle_{\text{s}} = 0 = {}_{\text{s}}\langle z; a_k|\hat{P}_k|z; a_k\rangle_{\text{s}} = \langle\hat{P}_k\rangle$, which is an expected result if we look at definition (28) and recall that $w_k = 0$. In these circumstances, it is possible to show, after some calculations, that

$$\begin{aligned}\sigma_X^{(k)}(z) &= {}_{\text{s}}\langle z; a_k|\hat{X}_k^2|z; a_k\rangle_{\text{s}}/{}_{\text{s}}\langle z; a_k|z; a_k\rangle_{\text{s}} \\ &= \left\{1 + 2\varrho_k(z) + [1 + \varrho_k(z)]\delta_k^{(s)}(z)\right\}\Delta_k^{(\text{H})}\end{aligned} \tag{63}$$

and

$$\begin{aligned}\sigma_P^{(k)}(z) &= {}_{\text{s}}\langle z; a_k|\hat{P}_k^2|z; a_k\rangle_{\text{s}}/{}_{\text{s}}\langle z; a_k|z; a_k\rangle_{\text{s}} \\ &= \left\{1 + 2\varrho_k(z) - [1 + \varrho_k(z)]\delta_k^{(s)}(z)\right\}\Delta_k^{(\text{H})}\end{aligned} \tag{64}$$

where the related factors are defined as

$$\delta_k^{(s)}(z) = 2\,\text{Re}\{\alpha_{k-1}^* z\mathcal{Z}_{k-2}/\alpha_k\} \tag{65}$$

with

$$\varrho_k(z) = \sum_{n=0}^{\infty}\left\{\frac{e_{2n}}{R(a_0)}\right\}p_n(z, a_k)\bigg/\sum_{n=0}^{\infty}p_n(z, a_k) \qquad \text{and} \qquad p_n(z, a_k) = \left|\frac{z^n}{h_{2n}(a_k)}\right|^2. \tag{66}$$

To conclude this section we can anticipate some general conclusions about the behaviour of the variances $\sigma_X^{(k)}(z)$ and $\sigma_P^{(k)}(z)$ which remain valid for any shape-invariant system just looking at the expressions obtained above. When $|z| \gg 1$ the presence of the energy factor e_{2n} losses importance in the sums of $p_n(z, a_k)$ and $\varrho_k(z)$ and thus we must have the factor $\varrho_k(z)$ going to a constant value independent of z. In these conditions the variances $\sigma_X^{(k)}(z)$ and $\sigma_P^{(k)}(z)$ are governed by a linear dependence on the factor $\delta_k^{(s)}(z)$. On the other hand, the positive defined values of the variances $\sigma_X^{(k)}(z)$ and $\sigma_P^{(k)}(z)$ impose restrictions on the range of allowed values of $\text{Re}\,z$.

4. Application to a potential with parameters related by a translation

Using the definition presented in the previous sections we illustrate in this and in the next sections the concept of generalized intelligent, coherent and squeezed states for two shape-invariant systems. Note that with the particularization of our generalized formalism for the harmonic oscillator potential case, the simplest shape-invariant potential, we are able to reproduce usual results obtained in the literature for purely coherent states [31], as well as for intelligent and purely squeezed states [32]. Therefore, as a first application of our generalized formalism we study the case of the Pöschl–Teller potential. Originally introduced in a molecular physics context [34], the Pöschl–Teller potential is closely related to several other potentials, widely used in molecular and solid state physics and, in addition, becomes the infinite square well in a limiting case. The partner potentials $V^{(\pm)}(x)$ in (3) for this system are obtained with the superpotential $W(x,a_1) = \sqrt{\hbar\Omega}\{(a_1+\gamma)\tan\rho x - (a_1-\gamma)\cot\rho x\}$ where a_1, ρ and γ are real constants [18]. The remainders [18] in the shape-invariant condition are given by $R(a_n) = 4\eta[2a_n+\eta]$, with the potential parameters related by $a_{n+1} = a_n + \eta$, where $\eta = \rho\sqrt{\hbar/(2M\Omega)}$. Inserting these results into (9) we get

$$e_n = 4\eta^2 n(n+\nu) \qquad \text{with} \quad \nu = 2a_1/\eta. \tag{67}$$

4.1. Results for generalizing factors independent of the potential parameters a_n

4.1.1. Purely coherent state. Using (67) we can prove that

$$\prod_{s=0}^{n-1}(e_n - e_s) = \prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right] = (2\eta)^{2n}\left[\frac{\Gamma(n+1)\Gamma(2n+\nu)}{\Gamma(n+\nu)}\right]. \tag{68}$$

Therefore, if we assume $\mathcal{Z}_k = 2\eta$ and use it together with result (68) in (52) we find for the expansion coefficient

$$h_n(a_k) = \sqrt{\frac{\Gamma(n+1)\Gamma(2n+\nu)}{\Gamma(n+\nu)}}. \tag{69}$$

Note at this point that if we choose $\nu = 1$ we get the simple expression

$$|z;a_k\rangle_c = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(2n)!}}|\Psi_n\rangle \tag{70}$$

found in [8] for the coherent state (51). Keeping the simplifying idea of this first example, we assume $\alpha_k = a_{k+1} - a_k = \eta$ that with the constant value of $\mathcal{Z}_k$ in (56) imply in $\delta_k^{(c)}(z) = 0$. Thus, by equations (54) and (55) we find $\sigma_X^{(k)} = \sigma_P^{(k)} = \Delta_k^{(\mathrm{H})} = 2\eta^4$, showing that, in this case, the purely coherent state does not present squeezing effects.

4.1.2. Purely squeezed state. In this case if we use the energy eigenvalue (67) we can prove that

$$\prod_{s=0}^{n-1}\sqrt{\left[\frac{e_{2n}-e_{2s}}{e_{2n}-e_{2s+1}}\right]} = \sqrt{\frac{\Gamma\left(2n+\frac{\nu}{2}\right)\Gamma\left(n+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(n+\frac{\nu}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}}. \tag{71}$$

With the simple choice $\mathcal{Z}_k = 2\eta$ and (71) in (61) and (60) we find the purely squeezed state expression

$$|z;a_k\rangle_s = \sum_{n=0}^{\infty}\sqrt{\frac{(4n+1)!!}{(2n+1)!}}(\eta z)^n|\Psi_{2n}\rangle \tag{72}$$

since we assume that $\nu = 2$. In these conditions, if we let $z = \mathrm{Re}\,z + \mathrm{i}\,\mathrm{Im}\,z = r\mathrm{e}^{\mathrm{i}\phi}$, we can evaluate the variances of the quadrature operators $\sigma_X^{(k)}(z)$ and $\sigma_P^{(k)}(z)$ using (63) and (64), respectively, with the factors (65) and (66), in this case, given by

$$p_n(r,a_k) = \frac{(4n+1)!!}{(2n+1)!}(\eta r)^{2n}, \qquad \frac{e_{2n}}{R(a_0)} = 4n(n+1), \qquad \delta_k^{(s)}(z) = 4\eta\,\mathrm{Re}\,z \tag{73}$$

where, for the sake of simplicity, we maintained the value $\alpha_k = a_{k+1} - a_k = \eta$, used in the coherent state case. In figure 1(*a*), we show a three-dimensional plot of the variance $\sigma_X^{(k)}(z)$-surface, measured in units of $\Delta_k^{(\mathrm{H})} = 2\eta^4$, as a function of $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ when $\eta = 0.2$. We observe that $\sigma_X^{(k)}(z)$ presents a depression region, like a valley, centred around $\mathrm{Re}\, z = -0.5$ and $\mathrm{Im}\, z = 0$ with radius $r \approx 4$, where its value increases nonlinearly from the axis $\mathrm{Re}\, z = -2.5$. Beyond the nonlinear *squeezing valley* the factor $\varrho_k(z)$ in (66) assumes the constant value $\varrho_k(z) \approx 100$, making the behaviour of $\sigma_X^{(k)}(z)$ to be basically governed by the plane function $\sigma_X^{(k)}(z) \approx 200 + 75\mathrm{Re}\, z$, which we call *squeezing plane*. To make clearer these observations, we show in figure 1(*b*) the contour plot of the variance $\sigma_X^{(k)}(z)$-surface on the $(\mathrm{Re}\, z, \mathrm{Im}\, z)$ plane in units of $\Delta_k^{(\mathrm{H})}$. The contour ranges from 0 to $320\Delta_k^{(\mathrm{H})}$ with the interval of $5\Delta_k^{(\mathrm{H})}$. Note that, in this case, it is not necessary to investigate the behaviour of the variance in $\hat{P}_k$ since $\sigma_P^{(k)}(r, \mathrm{Re}\, z) = \sigma_X^{(k)}(r, -\mathrm{Re}\, z)$. Because of this fact and the behaviour of $\sigma_X^{(k)}(z)$ presented in figures we can say that the particular influence of the system on the variances of the quadrature operators is concentrated in the squeezing valley region. The linear behaviour of the variances beyond this region is basically determined by the squeezed state characteristics. It is interesting to mention that another possible choices of the system parameters ν and η do not change qualitatively the results obtained. Only the extension of the valley and the slope of its lateral surface are sensible on these changes. As a final observation we highlight the symmetry of $\sigma_X^{(k)}(z)$ with $\mathrm{Im}\, z$ value and the range $-2.4 < \mathrm{Re}\, z < +2.4$ should be imposed on $\mathrm{Re}\, z$ values by the variances positive values.

To investigate the squeezing nature of the state (72) we can introduce a *variance deviation factor* defined as $\mathcal{D}_k(z) \equiv \sigma_X^{(k)}(z) - \sqrt{\sigma_X^{(k)}(z)\sigma_P^{(k)}(z)}$. Therefore, if $\mathcal{D}_k(z) > 0$ we have a *P*-squeezed state and if $\mathcal{D}_k(z) < 0$ the state will be *X*-squeezed. In figure 1(*c*) we show the contour plot of the $\mathcal{D}_k(z)$-surface on the $(\mathrm{Re}\, z, \mathrm{Im}\, z)$ plane in units of $\Delta_k^{(\mathrm{H})}$. The contour ranges from $-65\Delta_k^{(\mathrm{H})}$ to $280\Delta_k^{(\mathrm{H})}$ with the interval of $5\Delta_k^{(\mathrm{H})}$. It is evident from this figure that the state $|z; a_k\rangle_{\mathrm{s}}$ defined in (72) is *X*-squeezed when $\mathrm{Re}\, z < 0$ and *P*-squeezed when $\mathrm{Re}\, z > 0$.

4.2. Results for generalizing factors dependent on the potential parameters a_n

4.2.1. Purely coherent state. In order to take a little bit of our generalized study, we take into account the a_n-parameters translation relation to introduce a simple linear function

$$f(a_k; c, d) = ca_k + d \quad \text{yielding} \quad \prod_{s=0}^{n-1} f(a_{k+s}; c, d) = (c\eta)^n \left[\frac{\Gamma\left(n + k + \frac{\nu}{2} + \frac{d}{c\eta} - 1\right)}{\Gamma\left(k + \frac{\nu}{2} + \frac{d}{c\eta} - 1\right)} \right] \tag{74}$$

where c and d are constants. Therefore, if we define the generalizing functional with the form

$$\mathcal{Z}_k = \sqrt{f(a_2; 4, -4\eta) f(a_1; 4, 2\eta)} \qquad \text{then} \quad \prod_{s=0}^{n-1} \mathcal{Z}_{k+s} = (2\eta)^n \sqrt{\frac{\Gamma(2n+\nu)}{\Gamma(\nu)}} \tag{75}$$

and using (75) and (68) in (52) and (51) we obtain

$$|z; a_k\rangle_{\mathrm{c}} = \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+\nu)}{\Gamma(\nu)\Gamma(n+1)}} z^n |\Psi_n\rangle \tag{76}$$

that we can identify with the form found in [35] when we change $\nu \to \nu + 1$. With the choice (75) for the generalization factor $\mathcal{Z}_k$ we find the expression

$$\delta_k^{(c)}(z) = (\nu - 2)\left\{1 - \sqrt{\frac{(\nu-4)(\nu-3)}{(\nu-2)(\nu-1)}}\right\} \mathrm{Re}\, z^2 \tag{77}$$

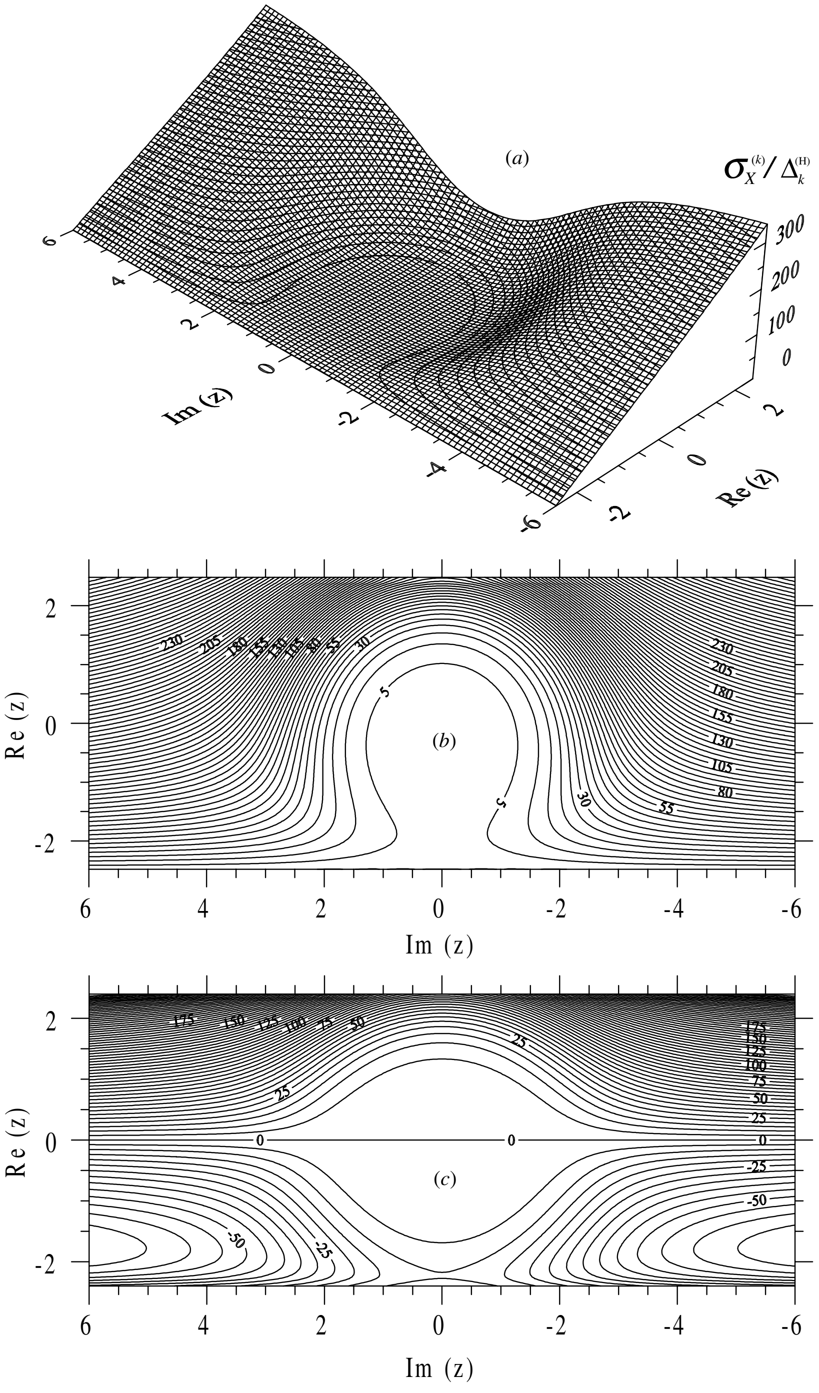

Figure 1. (*a*) The three-dimensional plot of the variance $\sigma_X^{(k)}(z)$-surface, measured in units of $\Delta_k^{(\mathrm{H})} = 2\eta^4$, as a function of Re z and Im z when $\eta = 0.2$. (*b*) The contour plot of $\sigma_X^{(k)}(z)$-surface on the (Re z, Im z) plane in units of $\Delta_k^{(\mathrm{H})}$. The contour ranges from 0 to $320\Delta_k^{(\mathrm{H})}$ with the interval of $5\Delta_k^{(\mathrm{H})}$. (*c*) The contour plot of the variance deviation $\mathcal{D}_k(z)$-surface on the (Re z, Im z) plane in units of $\Delta_k^{(\mathrm{H})}$. The contour ranges from $-65\Delta_k^{(\mathrm{H})}$ to $280\Delta_k^{(\mathrm{H})}$ with the interval of $5\Delta_k^{(\mathrm{H})}$.

for the squeezing factor (56) of the coherent state $|z; a_k\rangle_c$ since we keep the simplifying value $\alpha_k = a_{k+1} - a_k = \eta$ for the quadrature operators' generalizing factor. Note that with $\nu = 2$ we obtain a purely coherent state without squeezing effects since $\delta_k^{(c)} = 0$.

4.2.2. Purely squeezed state. In this case, with the help of the a_n-parameters translation relation we can show that the auxiliary function defined in (74) satisfies the relation

$$\prod_{s=1}^{n} f(a_{k+2s-2}; c, d) = (2c\eta)^n \left\{ \frac{\Gamma\left[n + \frac{1}{2}\left(k + \frac{\nu}{2} + \frac{d}{c\eta} - 1\right)\right]}{\Gamma\left[\frac{1}{2}\left(k + \frac{\nu}{2} + \frac{d}{c\eta} - 1\right)\right]} \right\}. \tag{78}$$

Therefore, if we use the result (78) and define the generalizing functional with the form

$$\mathcal{Z}_k = \sqrt{\frac{f(a_2; 1/2\eta, \nu/4)}{f(a_1; 1/2\eta, \nu/4)}} \qquad \text{we obtain} \qquad \prod_{s=1}^{n} \mathcal{Z}_{k+2s-2} = \sqrt{\frac{\Gamma\left(n + \frac{\nu}{2} + \frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(n + \frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2} + \frac{1}{2}\right)}} \tag{79}$$

that together with (71) in (61) gives for the purely squeezed state (60) the expression

$$|z; a_k\rangle_s = \sqrt{\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2} + \frac{1}{2}\right)}} \sum_{n=0}^{\infty} \sqrt{\frac{(2n-1)!!\Gamma\left(2n + \frac{\nu}{2} + \frac{1}{2}\right)}{n!\Gamma\left(2n + \frac{\nu}{2}\right)}} \left(\frac{z}{\sqrt{2}}\right)^n |\Psi_{2n}\rangle. \tag{80}$$

In this case, the factors (65) and (66) related with the variances of the quadrature operators, and responsible for the squeezing effects of $|z; a_k\rangle_s$, can be evaluated with the expressions

$$p_n(r, a_k) = \left\{ \frac{(2n-1)!!\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(2n + \frac{\nu}{2} + \frac{1}{2}\right)r^{2n}}{2^n n!\Gamma\left(\frac{\nu}{2} + \frac{1}{2}\right)\Gamma\left(2n + \frac{\nu}{2}\right)} \right\}, \qquad \frac{e_{2n}}{R(a_0)} = 4n(n+1),$$
$$\delta_k^{(s)}(z) = \sqrt{2}\,\mathrm{Re}\, z \tag{81}$$

where we kept the value $\alpha_k = a_{k+1} - a_k = \eta$, used before.

Figure 2 is the version of figure 1 for the squeezed state (80) calculated with $\nu = 2$. Note in figure 2(*a*) that, in this case, the nonlinear squeezing region, sensible to the system influence on the variances of the quadrature operators, shows a *bowl-shaped structure* centred around $\mathrm{Re}\, z = -0.1$ and $\mathrm{Im}\, z = 0$ with radius $r \approx 1$. Outside of the bowl region the factor $\varrho_k(z)$ goes to the constant value $\varrho_k(z) \approx 410$ and the behaviour of $\sigma_X^{(k)}(z)$ is basically governed by the squeezing plane function $\sigma_X^{(k)}(z) \approx 830 + 580\,\mathrm{Re}\, z$. We detach the stronger restriction on the $\mathrm{Re}\, z$ values imposed in this case. The behaviour of $\sigma_X^{(k)}(z)$ showing its asymmetry in $\mathrm{Re}\, z$ and symmetry in $\mathrm{Im}\, z$ makes clearer in figure 2(*b*) that shows the contour plot of $\sigma_X^{(k)}(z)$-surface on the $(\mathrm{Re}\, z, \mathrm{Im}\, z)$ plane in units of $\Delta_k^{(\mathrm{H})}$. The contour ranges from 0 to $1700\Delta_k^{(\mathrm{H})}$ with the interval of $20\Delta_k^{(\mathrm{H})}$. In figure 2(*c*) we show the contour plot of the variance deviation $\mathcal{D}_k(z)$-surface on the $(\mathrm{Re}\, z, \mathrm{Im}\, z)$ plane in units of $\Delta_k^{(\mathrm{H})}$. The contour ranges from $-340\Delta_k^{(\mathrm{H})}$ to $1500\Delta_k^{(\mathrm{H})}$ with the interval of $20\Delta_k^{(\mathrm{H})}$. In analogy with the other example, the X-squeezed nature of the state $|z; a_k\rangle_s$ when $\mathrm{Re}\, z < 0$ and its P-squeezed nature when $\mathrm{Re}\, z > 0$ is evident from the figure.

5. Application to a potential with parameters related by a scaling

One class of shape-invariant potentials is given by an infinite chain of reflectionless potentials $V_k^{(\pm)}(x)$, $(k = 0, 1, 2, \ldots)$, for which the associated superpotentials $W_k(x)$ satisfy the self-similar *ansatz* $W_k(x) = q^k W(q^k x)$, with $0 < q < 1$. These sets of partners potentials $V_k^{(\pm)}(x)$, also called self-similar potentials [36, 37], have an infinite number of bound states and their parameters related by a scaling: $a_n = q^{n-1} a_1$.

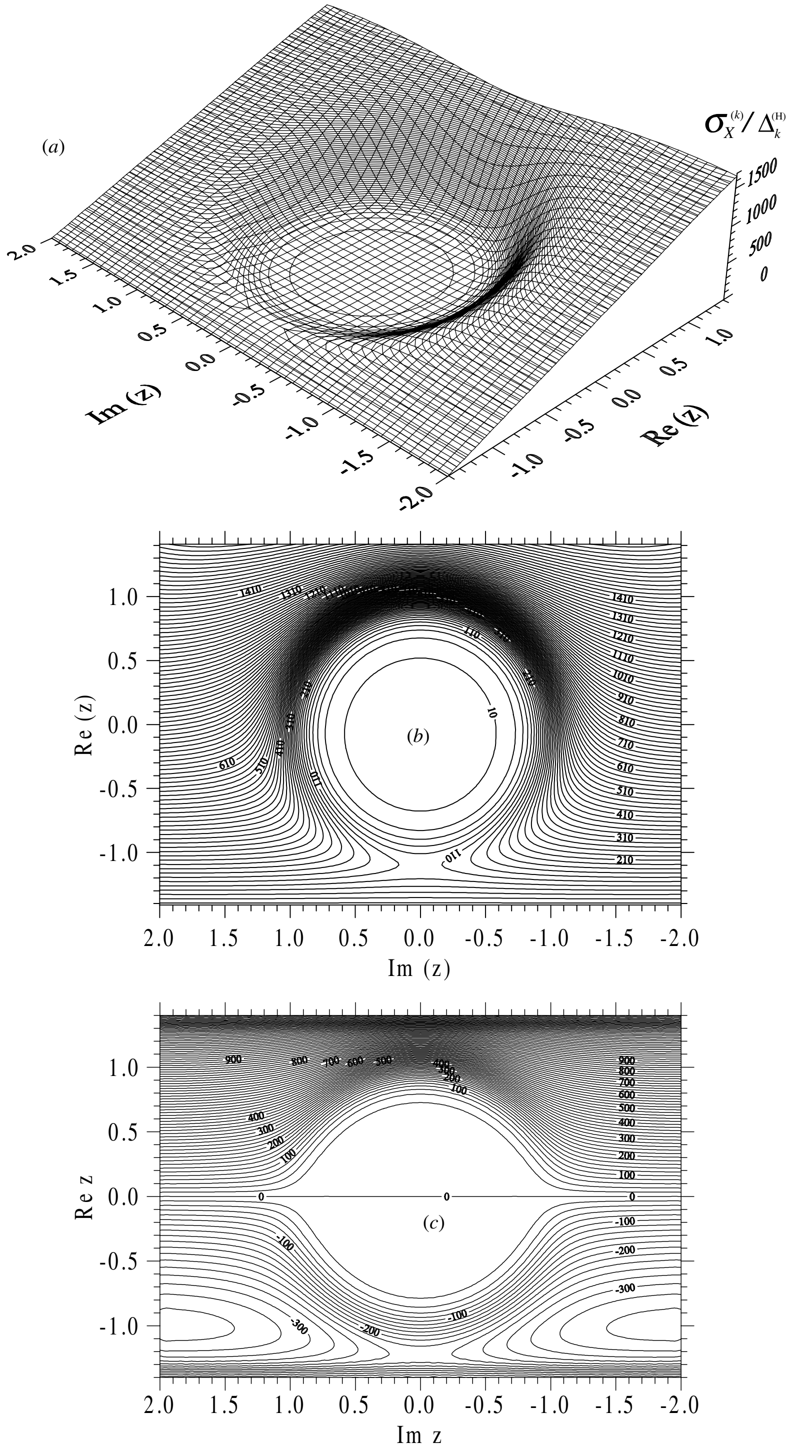

Figure 2. Same as figure 1, but for the squeezed state (80) calculated with $\nu = 2$. The contour plot of $\sigma_X^{(k)}(z)$-surface in (*b*) ranges from 0 to $1700\Delta_k^{(\mathrm{H})}$ with the interval of $20\Delta_k^{(\mathrm{H})}$. In (*c*) the contour plot of $\mathcal{D}_k(z)$-surface ranges from $-340\Delta_k^{(\mathrm{H})}$ to $1500\Delta_k^{(\mathrm{H})}$ with the interval of $20\Delta_k^{(\mathrm{H})}$.

5.1. Results for generalizing factors independent of the potential parameters a_n

5.1.1. Purely coherent state. Shape invariance of self-similar potentials was studied in detail in [38, 39]. In the simplest case studied the remainder in the shape invariance condition is given by $R(a_1) = ca_1$, where c is a constant. Using this result in (9) we obtain the energy factor of the system with the form

$$e_n = \left(\frac{1-q^n}{1-q}\right) R(a_1) \qquad \text{yielding}$$
$$\prod_{s=0}^{n-1}(e_n - e_s) = \prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right] = \left[\frac{R(a_1)}{1-q}\right]^n q^{n(n-1)/2}(q;q)_n \tag{82}$$

where the q-shifted factorial $(q;q)_n$ is defined as $(p;q)_0 = 1$ and $(p;q)_n = \prod_{k=0}^{n-1}(1-pq^k)$ with $n \in \mathbb{Z}$. Therefore, if we assume the simple choice $\mathcal{Z}_k = \sqrt{\sqrt{q}/(1-q)}$, a constant, and use it with the result of equation (82) in the expansion coefficient (52) we obtain the coherent state (51) with the form

$$|z; a_k\rangle_{\mathrm{c}} = \sum_{n=0}^{\infty} \frac{q^{-n^2/4}}{\sqrt{(q;q)_n}} \xi_1^n |\Psi_n\rangle \tag{83}$$

where $\xi_1 = z\big/\sqrt{R(a_1)}$. The result (83) was obtained in our previous paper [11] for the coherent states of the self-similar potentials. Assuming that $\alpha_k = a_{k+1}/a_k = q = \text{constant}$ and using this fact with the constant $\mathcal{Z}_k$-value in (56) we conclude that $\delta_k^{(c)}(z) = 0$. Thus, by equations (54) and (55) we find $\sigma_X^{(k)} = \sigma_P^{(k)} = \Delta_k^{(\mathrm{H})} = \frac{1}{2}qR(a_1)$, showing that in this case the purely coherent state does not present squeezing effects.

5.1.2. Purely squeezed state. Using the expression of e_n for self-similar potentials we can also prove that

$$\prod_{s=0}^{n-1}\sqrt{\left[\frac{e_{2n}-e_{2s}}{e_{2n}-e_{2s+1}}\right]} = \sqrt{\frac{(q^2;q^2)_n}{q^n(q;q^2)_n}}. \tag{84}$$

Therefore, if we assume now the constant value $\mathcal{Z}_k = \sqrt{q}$ we find for the expansion coefficient (61) and the purely squeezed state (60) the expressions

$$h_{2n}(a_k) = \sqrt{\frac{(q^2;q^2)_n}{q^{2n}(q;q^2)_n}} \qquad \text{and} \qquad |z;a_k\rangle_{\mathrm{s}} = \sum_{n=0}^{\infty}\sqrt{\frac{(q;q^2)_n}{(q^2;q^2)_n}}(qz)^n|\Psi_{2n}\rangle. \tag{85}$$

In this case, the factors (65) and (66) related with the variances of the quadrature operators can be evaluated with

$$p_n(r,a_k) = \left[\frac{(q;q^2)_n}{(q^2;q^2)_n}\right](qr)^{2n}, \qquad \frac{e_{2n}}{R(a_0)} = \frac{q(1-q^{2n})}{1-q}, \qquad \delta_k^{(s)}(z) = 2\sqrt{q}\,\mathrm{Re}\,z \tag{86}$$

where we kept the value $\alpha_k = a_{k+1}/a_k = q$ used in the coherent state case.

In figure 3, we show the three-dimensional plot of the variance $\sigma_X^{(k)}(z)$-surface, measured in units of $\Delta_k^{(\mathrm{H})}$, as a function of $\mathrm{Re}\,z$ and $\mathrm{Im}\,z$ when the scaling parameter has the values $q = 0.60$ and $q = 0.99$. For the lower value of q we observe that $\sigma_X^{(k)}(z)$ presents a depression region centred around $\mathrm{Im}\,z$-axis and with radius $r \approx 1.8$ where its value increases nonlinearly from the axis $\mathrm{Re}\,z = -1.0$. Beyond this *squeezing valley* the presence of the energy system term $e_{2n}/R(a_0)$ in expression (66), which defines $\varrho_k(z)$, is unrelevant making this factor to assume the constant value $\varrho_k(z) \approx q/(1-q) = 1.5$. Thus, by (63) the behaviour of $\sigma_X^{(k)}(z)$ is

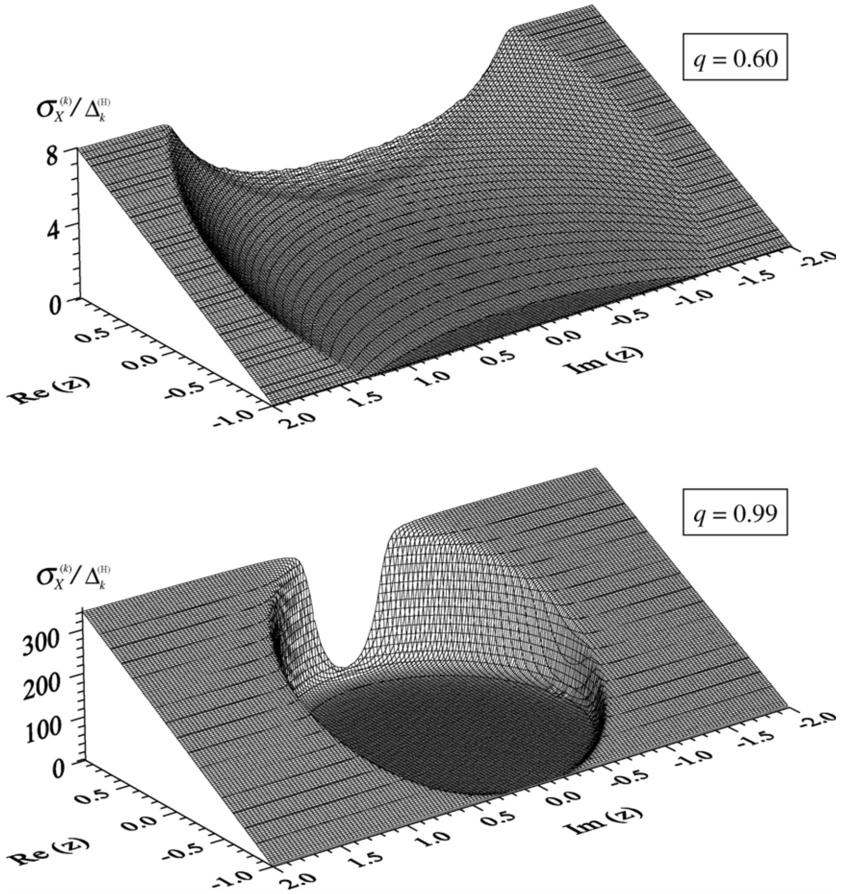

Figure 3. The three-dimensional plot of the variance $\sigma_X^{(k)}(z)$-surface, measured in units of $\Delta_k^{(\mathrm{H})} = \frac{1}{2}qR(a_1)$, as a function of $\mathrm{Re}\,z$ and $\mathrm{Im}\,z$ for the scaling parameter values $q = 0.60$ and 0.99.

basically governed by the squeezing plane function $\sigma_X^{(k)}(z) \approx 4(1+\mathrm{Re}\,z)$. Taking into account this result and the result for the higher value $q = 0.99$ we conclude that the scaling parameter q has a fundamental relevance on the variance behaviour. For $q = 0.99$ the nonlinear squeezing region is more restricted in size and well defined in shape, like a bowl cut by the squeezing plane $\sigma_X^{(k)}(z) \approx 170(1 + \mathrm{Re}\,z)$, defined when $r > 1$ and the factor $\varrho_k(z)$ goes to the constant value $\varrho_k(z) \approx q/(1-q) = 99$. As in the other cases, taking the variance deviation factor $\mathcal{D}_k(z)$ it is possible to conclude that the state $|z; a_k\rangle_{\mathrm{s}}$ defined in (85) is X-squeezed when $\mathrm{Re}\,z < 0$ and P-squeezed when $\mathrm{Re}\,z > 0$. On the other hand, the symmetry relation $\sigma_P^{(k)}(r, \mathrm{Re}\,z) = \sigma_X^{(k)}(r, -\mathrm{Re}\,z)$ remains valid, which makes it unnecessary to discuss the behaviour of the variance $\sigma_P^{(k)}$ in the other quadrature operator. In this case, the positive defined value of the variance restricts $\mathrm{Re}\,z$ to the range $-1.0 < \mathrm{Re}\,z < +1.0$.

5.2. *Results for generalizing factors dependent on the potential parameters a_n*

5.2.1. Purely coherent state. As in the potential parameter translation case, just to take a little bit of our generalized approach for this kind of potential system, let us assume

$$\mathcal{Z}_k = R(a_1) \qquad \text{yielding} \qquad \prod_{s=0}^{n-1} \mathcal{Z}_{k+s} = [R(a_1)]^n q^{n(n-1)/2}. \tag{87}$$

Substituting this result and equation (82) into (52) and (51) we find

$$h_n(a_k) = \sqrt{\frac{(q;q)_n}{[R(a_1)(1-q)]^n q^{n(n-1)/2}}} \qquad \text{and} \qquad |z; a_k\rangle_{\mathrm{c}} = \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{\sqrt{(q;q)_n}} \xi_2^n |\Psi_n\rangle \tag{88}$$

where $\xi_2 = z\sqrt{R(a_1)(1-q)}/\sqrt{q}$. This expression was found in our previous paper [12] as a possible coherent state for self-similar potentials. With choice (87) for the generalization factor $\mathcal{Z}_k$, we find the expression $\delta_k^{(c)}(z) = -q^{-3/2}\,\mathrm{Re}\{\xi_2^2\}$ for the squeezing factor (56) of the coherent state $|z; a_k\rangle_{\mathrm{c}}$, since we keep the simplifying value $\alpha_k = a_{k+1}/a_k = q$ for the quadrature operators generalizing factor. Note that in the limit $q \to 1$ we have $\delta_k^{(c)} \to 0$ and we obtain again a purely coherent state without squeezing effects.

5.2.2. Purely squeezed state. With the form (87) for the generalizing factor $\mathcal{Z}_k$ we obtain

$$\prod_{s=1}^{n} \mathcal{Z}_{k+2s-2} = [R(a_1)]^n q^{n(n-1)} \tag{89}$$

and using (84) and (89) in (61) and (60) we conclude that

$$h_{2n}(a_k) = \sqrt{\frac{(q^2;q^2)_n}{q^n(q;q^2)_n}}\Bigg/ [R(a_1)]^n q^{n(n-1)} \qquad \text{and}$$

$$|z; a_k\rangle_{\mathrm{s}} = \sum_{n=0}^{\infty} q^{n^2} \sqrt{\frac{(q;q^2)_n}{(q^2;q^2)_n}} \zeta_2^n |\Psi_{2n}\rangle \tag{90}$$

where $\zeta_2 = R(a_1)z/\sqrt{q}$. In this case, the factors (65) and (66), related with the variances of the quadrature operators, can be evaluated with

$$p_n(z, a_k) = \left[\frac{q^{2n^2}(q;q^2)_n}{(q^2;q^2)_n}\right] |\zeta_2|^{2n}, \qquad \frac{e_{2n}}{R(a_0)} = \frac{q(1-q^{2n})}{1-q},$$

$$\delta_k^{(s)}(z) = 2R(a_1)q^{-2}\,\mathrm{Re}\, z \tag{91}$$

and $\alpha_k = a_{k+1}/a_k = q$.

Figure 4 is the version of figure 3 for the squeezed state (90), calculated with $R(a_1) = 1$ and for the scaling parameter values $q = 0.800, 0.990$ and 0.999. Note that, in this case, the nonlinear squeezing region of the variances shows a more sensibility to higher values of the scaling parameter. Only for q values very close to unity the three-dimensional plot of the variance $\sigma_X^{(k)}(z)$-surface as a function of $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ shows a restricted and well-defined shape. For lower values of q the restrictions on the $\mathrm{Re}\, z$ values are stronger. On the other hand, when $q \to 1$ the squeezing plane of $\sigma_X^{(k)}(z)$ that cut the bowl-shaped structure, defined when $r > 1.5$ and $\varrho_k(z) \to 95$, is obtained by equation $\sigma_X^{(k)}(z) \approx 190(1 + \mathrm{Re}\, z)$. As in other examples, the X-squeezed and P-squeezed nature of the state (90) when $\mathrm{Re}\, z < 0$ and $\mathrm{Re}\, z > 0$, respectively, is still valid. The same we can say about the symmetry relation $\sigma_P^{(k)}(r, \mathrm{Re}\, z) = \sigma_X^{(k)}(r, -\mathrm{Re}\, z)$.

To close these applications for self-similar potentials, note that in the limit of

$$\lim_{q\to 1} e_n = \lim_{q\to 1} \left(\frac{1-q^n}{1-q}\right) R(a_1) \to n R(a_1) \qquad \text{and} \qquad \lim_{q\to 1} (q;q)_n \to n!, \tag{92}$$

implying that the results obtained must reduce to those of the harmonic oscillator potential case.

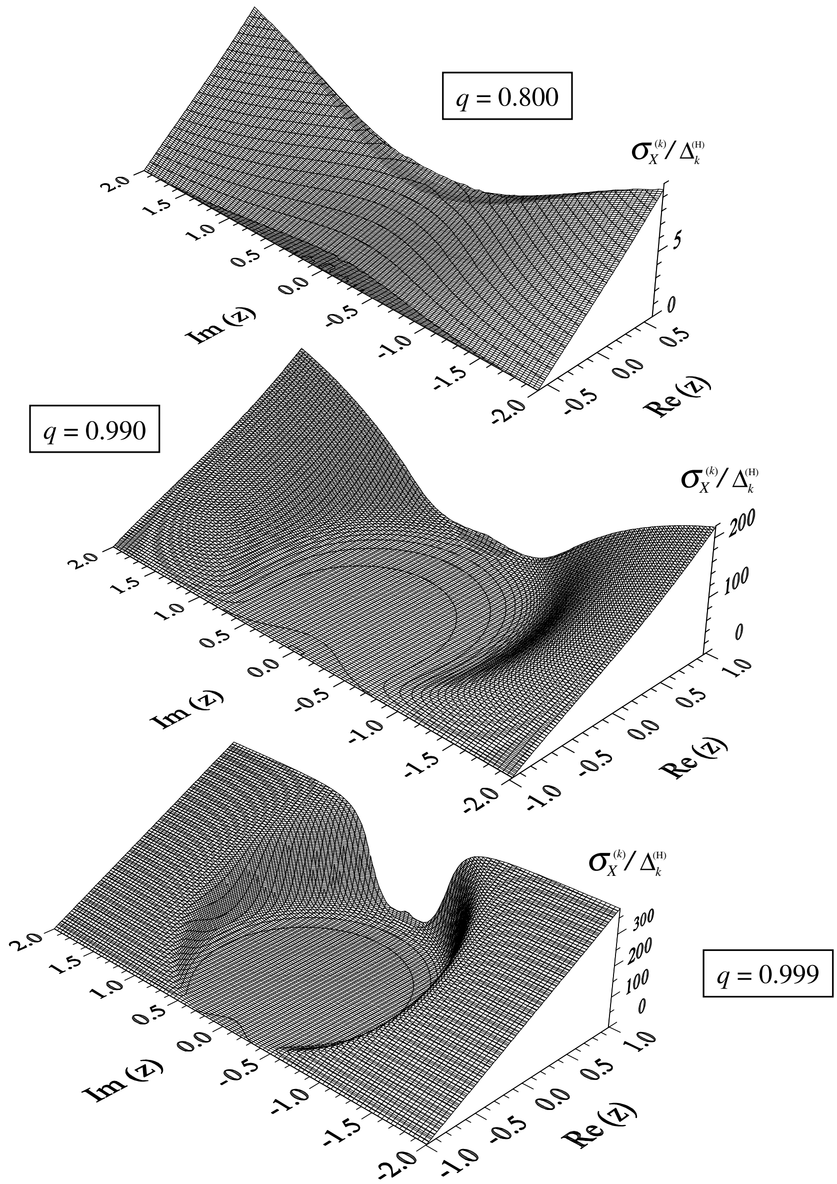

Figure 4. Same as figure 3, but for the squeezed state (90) calculated with $R(a_1) = 1$ and with the scaling parameter values $q = 0.800$, 0.990 and 0.999.

6. Final remarks

In this paper, using an algebraic approach, we constructed generalized Robertson intelligent states for shape-invariant systems. This generalization based on the introduction of quadrature operators and factors which depend on the potential parameters makes it possible to investigate coherency and squeezing effects in the shape-invariant systems and evaluate the dependence of these effects in the different choices of the generalizations factors for shape-invariant systems with potential parameters related by a translation and by a scaling function. We showed that the squeezing effects, when analysed in terms of the real and imaginary parts of the z-parameter, can be separated in two regions with very different behaviours. An internal region, called *squeezing valley*, which presents a nonlinear behaviour very sensible to the system characteristics, and an external region, called *squeezing plane*, which presents a linear dependence in Re z, determined basically for the state generalizing factor $\mathcal{Z}_k$.

For coherent states in general, and for the squeezed states of harmonic oscillator systems in particular, we showed that, with an adequate choice of the generalizing factors, it is possible to reproduce results already known in the literature.

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